Pure Core 3 Past Paper Questions Pack A

Taken from MAP1

January 2001

2 (a) Sketch on one pair of axes the graphs of

$$y = 6 - x \text{ and } y = \ln x \,.$$ *(1 mark)*

(b) Hence state the number of roots of the equation

$$6 - x = \ln x \,.$$ *(1 mark)*

(c) By considering values of the function f, where

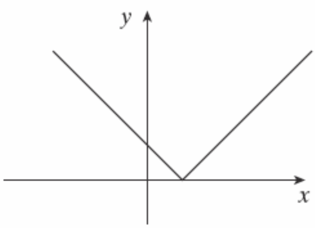
$$f(x) = 6 - x - \ln x \,,$$

(i) show that the equation in part (b) has a root α such that

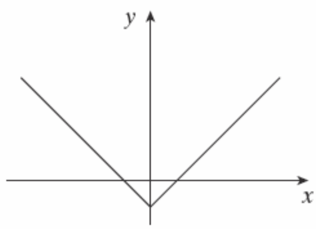
$$4 < \alpha < 5 \,,$$ *(2 marks)*

(ii) determine whether α is closer to 4 or to 5. *(2 marks)*

4



$y = f(x)$



$y = g(x)$

The diagrams show the graphs of $y = f(x)$ and $y = g(x)$, where the functions f and g are defined on the domain of all real numbers by

$$f(x) = |x - 2| \text{ and } g(x) = |x| - 2.$$

(a) Describe the geometrical transformations by which each of the above graphs can be obtained from the graph of $y = |x|$. *(2 marks)*

(b) Sketch the graph of $y = f(x) - g(x)$. *(2 marks)*

(c) (i) State whether the function f has an inverse function.

(ii) State whether the function g is even, odd or neither.

(iii) Give the range of the function h, where $h(x) = f(x) - g(x)$. *(3 marks)*

(d) Solve the following inequalities:

(i) $f(x) < 2$,

(ii) $g(x) < 2$,

(iii) $f(x) > g(x)$. *(4 marks)*

June 2001

7 (a) Solve the inequality

$$|x-3| > 1.$$ *(3 marks)*

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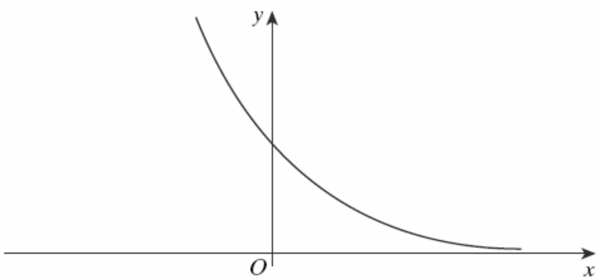


The diagram shows the graph of $y = \mathrm{f}(x)$, where f is defined for all real numbers by

$$\mathrm{f}(x) = 2\mathrm{e}^{-x}.$$

(a) Describe a sequence of geometrical transformations by which the above graph can be obtained from the graph of $y = \mathrm{e}^x$. *(3 marks)*

(b) Copy the above diagram and sketch on the same axes the graph of

$$y = \mathrm{f}^{-1}(x).$$ *(2 marks)*

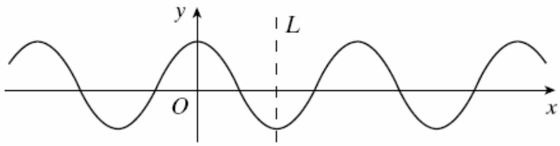
(c) Find an expression for $\mathrm{f}^{-1}(x)$. *(3 marks)*

(d) State the domain and range of f^{-1}. *(2 marks)*

(e) At time t hours after an injection, a hospital patient has $\mathrm{f}(t)$ milligrams per litre of a certain drug in his blood. Find the time after the injection at which the patient has 0.5 milligrams per litre of the drug in his blood. *(3 marks)*

January 2002

4 The diagram shows a sketch of the graph of $y = \cos 2x$ with a line of symmetry L.



(a) (i) Describe the geometrical transformation by which the graph of

$$y = \cos 2x$$

can be obtained from that of $y = \cos x$. *(2 marks)*

(ii) Write down the equation of the line L. *(1 mark)*

The function f is defined for the restricted domain $0 \leqslant x \leqslant \frac{\pi}{2}$ by

$$f(x) = \cos 2x.$$

(b) (i) State the range of the function f. *(1 mark)*

(ii) Write down the domain and range of the inverse function f^{-1}, making it clear which is the domain of f^{-1} and which is its range. *(2 marks)*

(iii) Sketch the graph of $y = f^{-1}(x)$. *(2 marks)*

The function g is defined for all real numbers by

$$g(x) = |x|.$$

(c) (i) Write down an expression for $gf(x)$. *(1 mark)*

(ii) Sketch the graph of $y = gf(x)$. *(2 marks)*

6 A graph has equation $y = (e^x - 1)(e^x - 2)$. The following correct reasoning is used to find $\frac{dy}{dx}$.

$$y = (e^x - 1)(e^x - 2)$$
$$\Rightarrow \quad y = e^{2x} - 3e^x + 2$$
$$\Rightarrow \quad \frac{dy}{dx} = 2e^{2x} - 3e^x$$
$$\Rightarrow \quad \frac{dy}{dx} = e^x(2e^x - 3)$$

(a) Using these results,

(i) give a reason why the graph has only one stationary point, *(2 marks)*

(ii) find the coordinates of the stationary point, *(3 marks)*

(iii) find the value of $\frac{d^2y}{dx^2}$ at the stationary point, and hence determine whether the stationary point is a maximum or a minimum. *(4 marks)*

(b) (i) Show that the graph intersects the x-axis when $x = 0$ and when $x = \ln 2$. *(2 marks)*

(ii) Show that the area of the region below the x-axis enclosed by the graph and the x-axis is

$$\frac{3}{2} - 2\ln 2.$$ *(5 marks)*

June 2002

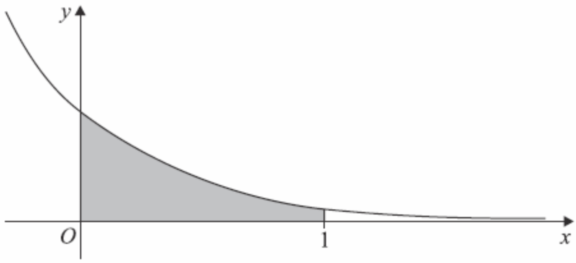
3 (a) Differentiate:

(i) $2x^{\frac{1}{2}}$;

(ii) $\ln(x + 1)$. *(3 marks)*

(b) Hence show that $\int_1^4 \left(x^{-\frac{1}{2}} + \frac{1}{x+1}\right) dx = 2 + \ln \frac{5}{2}$. *(5 marks)*

4



The diagram shows the graph of

$$y = e^{-2x}.$$

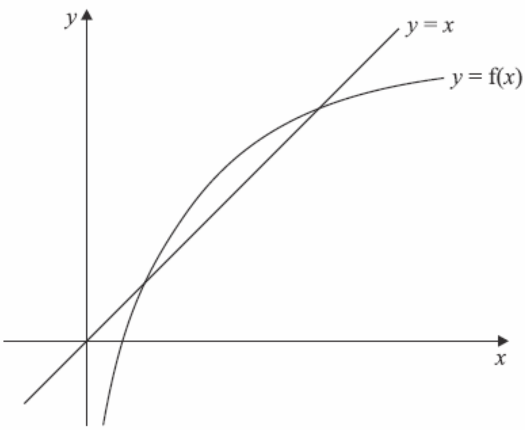
(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. *(3 marks)*

(b) (i) Find $\int y \, dx$. *(2 marks)*

(ii) Hence show that the area of the region shaded on the diagram is

$$\frac{e^2 - 1}{2e^2}.$$ *(3 marks)*

7



The diagram shows the graphs of $y = x$ and $y = f(x)$.

(a) (i) Describe the geometrical transformation by which the graph of $y = f^{-1}(x)$ can be obtained from the graph of $y = f(x)$. *(1 mark)*

(ii) Copy the above diagram and sketch on the same axes the graph of

$$y = f^{-1}(x).$$ *(2 marks)*

(b) The function f is defined for $x > 0$ by

$$f(x) = 3 \ln x.$$

(i) Describe the geometrical transformation by which the graph of $y = f(x)$ can be obtained from the graph of $y = \ln x$. *(2 marks)*

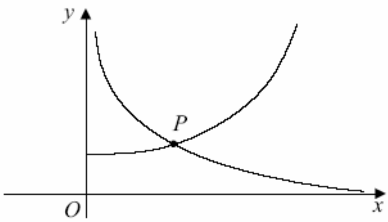
(ii) Find an expression for $f^{-1}(x)$. *(3 marks)*

November 2002

1 The diagram shows the graphs of

$$y = x^2 + 1 \text{ and } y = \frac{1}{x}$$

for $x > 0$. The graphs intersect at the point P.



(a) Show that the x-coordinate of P satisfies the equation

$$x^3 + x - 1 = 0.$$ *(2 marks)*

(b) Show that the x-coordinate of P lies between 0.6 and 0.7. *(3 marks)*

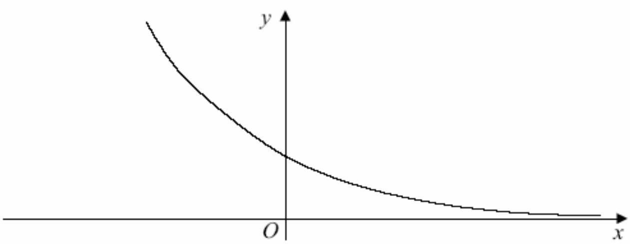
3 (a) Show that $\int_1^4 x^{\frac{3}{2}}\,dx = \frac{62}{5}$. *(4 marks)*

(b) Find the value of

$$\int_2^{18} \frac{1}{2x}\,dx,$$

giving your answer in the form $\ln n$. *(4 marks)*

7 The diagram shows the graph of $y = 2e^{-x}$.



(a) Describe a series of geometrical transformations by which the graph of $y = 2e^{-x}$ can be obtained from that of $y = e^x$. *(3 marks)*

(b) The function f is defined **for the restricted domain $x \geqslant 0$** by

$$f(x) = 2e^{-x}.$$

(i) State the range of the function f. *(2 marks)*

(ii) State the domain and range of the inverse function f^{-1}. *(2 marks)*

(iii) Find an expression for $f^{-1}(x)$. *(3 marks)*

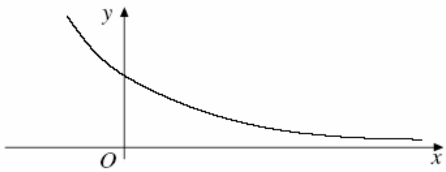
(iv) State, giving a reason, whether

$$x > \ln 2 \Rightarrow f(x) < 1.$$ *(2 marks)*

January 2003

5 (a) The diagram shows the graph of $y = \mathrm{f}(x)$, where the function f is defined for all values of x by

$$\mathrm{f}(x) = 5\mathrm{e}^{-x}.$$



(i) Write down the coordinates of the point where the graph intersects the y-axis. *(1 mark)*

(ii) State the range of the function f. *(1 mark)*

(iii) Find the value of $\mathrm{f}(\ln 6)$, giving your answer as a fraction. *(2 marks)*

(b) The function g is defined for all values of x by

$$\mathrm{g}(x) = x + 10.$$

(i) Show that $\mathrm{gf}(x) = 5(\mathrm{e}^{-x} + 2)$. *(1 mark)*

(ii) State the range of the function gf. *(1 mark)*

(iii) Sketch the graph of $y = \mathrm{gf}(x)$. *(2 marks)*

(iv) Show that $\mathrm{gf}(x) = 11 \Rightarrow x = \ln 5$. *(3 marks)*

(c) A dish of water is left to cool in a room where the temperature is 10 °C. At time t minutes, where $t \geqslant 0$, the temperature of the water in degrees Celsius is

$$5(\mathrm{e}^{-t} + 2).$$

(i) State the temperature of the water at time $t = 0$. *(1 mark)*

(ii) Calculate the time at which the temperature of the water reaches 11 °C. Give your answer to the nearest tenth of a minute. *(2 marks)*

6 The function f is defined for $x \geqslant 0$ by

$$f(x) = x^{\frac{1}{2}} + 2.$$

(a) (i) Find $f'(x)$. *(2 marks)*

(ii) Hence find the gradient of the curve $y = f(x)$ at the point for which $x = 4$. *(1 mark)*

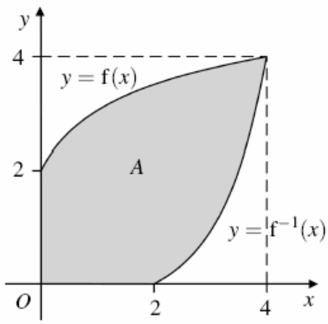
(b) (i) Find $\int f(x)\,dx$. *(3 marks)*

(ii) Hence show that $\int_0^4 f(x)\,dx = \frac{40}{3}$. *(2 marks)*

(c) Show that $f^{-1}(x) = (x-2)^2$. *(2 marks)*

(d) The diagram shows a symmetrical shaded region A bounded by:

parts of the coordinate axes;
the curve $y = f(x)$ for $0 \leqslant x \leqslant 4$; and
the curve $y = f^{-1}(x)$ for $2 \leqslant x \leqslant 4$.



(i) Write down the equation of the line of symmetry of A. *(1 mark)*

(ii) Calculate the area of A. *(4 marks)*

June 2003

4 It is given that x satisfies the equation

$$2\cos^2 x = 2 + \sin x.$$

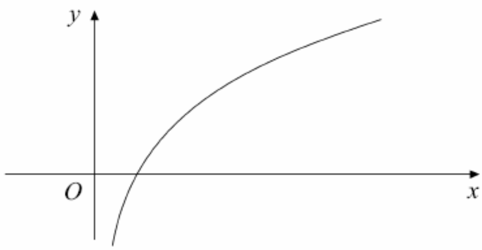
(a) Use an appropriate trigonometrical identity to show that

$$2\sin^2 x + \sin x = 0.$$ *(2 marks)*

(b) Solve this quadratic equation and hence find all the possible values of x in the interval $0 \leqslant x < 2\pi$. *(6 marks)*

5 The diagram shows the graph of $y = \mathrm{f}(x)$, where f is defined for $x > 0$ by

$$\mathrm{f}(x) = 2 + \ln x.$$



(a) (i) Differentiate $\mathrm{f}(x)$ to find $\mathrm{f}'(x)$. *(1 mark)*

(ii) Find the gradient of the curve at the point where $x = \mathrm{e}$. *(1 mark)*

(b) Describe the geometrical transformation by which the graph of $y = 2 + \ln x$ can be obtained from the graph of $y = \ln x$. *(2 marks)*

(c) (i) State the range of the function f. *(1 mark)*

(ii) State the domain and range of the inverse function f^{-1}. *(2 marks)*

(iii) Find an expression for $\mathrm{f}^{-1}(x)$. *(3 marks)*

(d) The function g is defined for all x by

$$\mathrm{g}(x) = \mathrm{e}x^3.$$

Show that:

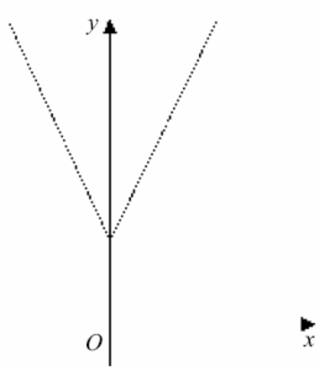
(i) $\mathrm{fg}(x) = 3(1 + \ln x)$; *(3 marks)*

(ii) $\mathrm{fg}(x) = 9 \Rightarrow x = \mathrm{e}^2$. *(2 marks)*

November 2003

7 The diagram shows the graph of

$$y = 2|x| + 1.$$



(a) Copy the diagram and, on the same pair of axes, sketch the graph of

$$y = |2x + 1|.$$ *(2 marks)*

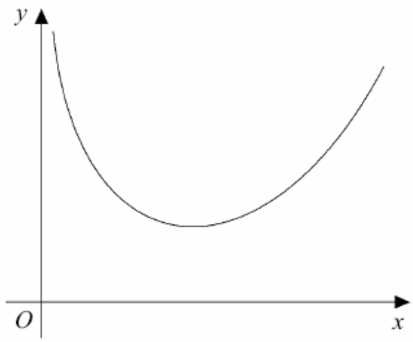
(c) Find the full solution set for the inequality

$$|2x + 1| < 2|x| + 1.$$ *(1 mark)*

January 2004

5 The diagram shows a sketch of the graph of

$$y = e^{2x} + 2x^{-1} \qquad \text{for } x > 0.$$



(a) Find $\dfrac{dy}{dx}$. *(3 marks)*

(b) Show that, at the stationary point on the graph, $x^2e^{2x} = 1$. *(3 marks)*

(c) Deduce that, at the stationary point,

$$xe^x = 1$$

and hence

$$\ln x + x = 0.$$ *(3 marks)*

(d) Show that the equation

$$\ln x + x = 0$$

has a root between 0.5 and 0.6. *(3 marks)*

(e) Find $\int (e^{2x} + 2x^{-1})dx$. *(3 marks)*

6 (a) The functions f and g are defined by:

$$f(x) = \sqrt{x} \qquad \text{for } x \geqslant 0;$$

$$g(x) = x - 1 \qquad \text{for all values of } x.$$

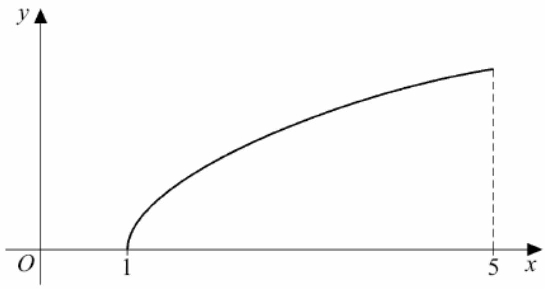
(i) Write down expressions for $fg(x)$ and $gf(x)$. *(2 marks)*

(ii) Verify that

$$x = 1 \Rightarrow fg(x) = gf(x).$$ *(1 mark)*

(b) The diagram shows the graph of $y = h(x)$, where the function h is defined **for the domain $1 \leqslant x \leqslant 5$** by

$$h(x) = \sqrt{x - 1}.$$



(i) Describe the transformation by which the graph of $y = \sqrt{x-1}$ can be obtained from the graph of $y = \sqrt{x}$. *(2 marks)*

(ii) Write down the range of the function h. *(1 mark)*

(iii) Write down the domain and range of the inverse function h^{-1}. *(2 marks)*

(iv) Find an expression for $h^{-1}(x)$. *(3 marks)*

June 2004

3 (a) Show that the equation

$$2x^{\frac{3}{2}} - 9x + 6 = 0$$

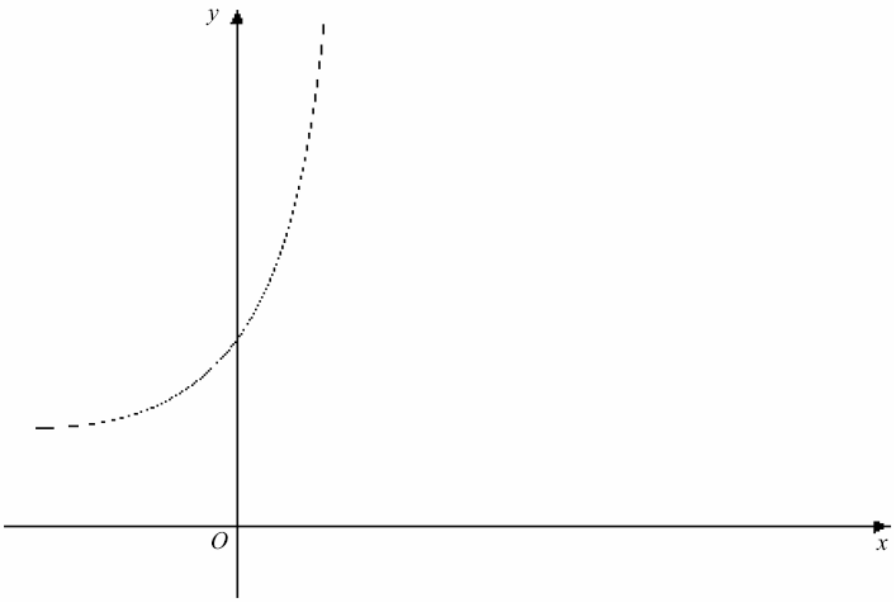
has a root between 0 and 1. *(3 marks)*

7 (a) (i) Find $\int (e^{2x}+1)\,dx$. *(3 marks)*

(ii) Hence show that $\int_0^{\ln 2} (e^{2x}+1)\,dx = \frac{3}{2} + \ln 2$. *(3 marks)*

(b) The diagram shows the graph of

$$y = e^{2x} + 1.$$



Find the y-coordinate of the point where the graph intersects:

(i) the y-axis; *(1 mark)*

(ii) the line $x = \ln 2$. *(2 marks)*

(c) The function f is defined **on the restricted domain $0 \leqslant x \leqslant \ln 2$** by

$$f(x) = e^{2x} + 1.$$

(i) Find the range of the function f. *(1 mark)*

(ii) On one pair of axes sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$. *(2 marks)*

(iii) Find an expression for $f^{-1}(x)$. *(3 marks)*